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Kernel Estimates of the Derivative of Regression Curves

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ABSTRACT

In nonparametric regression estimation, the estimation of the derivatives of regression curve is already investigated. As the domain of density function is bounded, it is known that the kernel regression estimator of Mack and Müller (1989) will also encounter the problem of the boundary effects for the estimation of the derivatives. In order to improve the boundary effects and bias reduction, one follow the idea of the minimizing quadratic form to construct a new kernel regression estimator in this paper. The new proposed estimator, the compact form of the asymptotic bias, the asymptotic variance and some properties are given. Besides, the proposed estimator will not produce the boundary effects and be improved the kernel regression estimator of Mack and Müller (1989) as above, its convergence rate is achieved $O(n^{-\frac{p+1-d}{2p+3}})$, for $d = 1,2,\ldots,p \geq 1$.

Key words: kernel regression estimator, boundary effects, derivatives, mean integrated square error, bias reduction.

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1. Introduction

Let $(X_1, Y_1),....,(X_n, Y_n)$ be a sequence of independent and identically distributed (IID) random variables with a common unknown joint probability density function $f(x, y)$. In the literature, various estimators for $m^{(d)}(x)$, based on a random sample $(X_1, Y_1),....,(X_n, Y_n)$, have been proposed and their properties are studied, for example, the books of Wand and Jones (1995), Simonoff (1996), Silverman (1986), Fan and Gijbels (1996) and Prakasa Rao (1983). A kernel regression estimate for $m^{(d)}$ is defined as follows:

$$\tilde{m}_1^{(d)}(x;h) = \frac{1}{nh^{d+1}}\sum_{j=1}^{n} K^{(d)}\left\{\frac{x-X_j}{h}\right\}\frac{Y_j}{f(X_j)} \quad , \tag{1.1}$$

where K is a symmetric kernel function and $h=h(n)$ is called the bandwidth or smoothing parameter. Here $\tilde{m}_1^{(d)}$ is d-times derivative of $\tilde{m}_1$, for all $d \geq 1$.

This estimator (1.1) is called the kernel regression estimator of Mack and Müller (1989) as $d \geq 1$. The studied of $m^{(d)}$ can also refer to, for examples, Gasser and Müller (1984), Ruppert and Wand (1994), for $d \geq 1$. When point x near the boundary of their support, the kernel regression estimator (KRE) (1.1) has been suffered to a seriously problem of boundary effects. About some boundary modification methods have been proposed by, for examples, Fan and Gijbels (1992), Fan (1993) and Müller (1993).

In this paper, researcher will relax the constraint that the kernel density function has to be a bona fide density, and propose a new KRE that it does not encounter the boundary effects. In this paper, proposed method is used the nice property of minimizing quadratic form (or least squares method). In order to get the new KRE for estimating $m^{(d)}(x)$, researcher first introduce the minimizing quadratic form (MQF) which is given by

$$\min_{\vec{\alpha}} Q(\vec{\alpha}) = \min_{\vec{\alpha}} (\vec{t} - A\vec{\alpha})'(\vec{t} - A\vec{\alpha}) \tag{1.2}$$

where A is an $(p+1)\times(p+1)$ symmetric matrix of constants, $\vec{t}$ is an $(p+1)\times 1$ vector of constants, and $\vec{\alpha}$ is an $(p+1)\times 1$ vector of unknown parameters. Here A and $\vec{t}$ are defined as follows:

$$A = \begin{bmatrix} S_0 & S_1 & S_2 & \cdots & S_P \\ S_1 & S_2 & S_3 & \cdots & S_{P+1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ S_P & S_{P+1} & S_{P+2} & \cdots & S_{2P} \end{bmatrix},$$

$$S_j = \frac{1}{h}\int K\left\{\frac{y-x}{h}\right\}\left(\frac{y-x}{h}\right)^j dy \ , \tag{1.3}$$

$$t_k = \frac{1}{nh}\sum_{j=1}^{n}\left(\frac{X_j-x}{h}\right)^k K\left\{\frac{X_j-x}{h}\right\}\frac{Y_j}{f(X_j)} \ , \tag{1.4}$$

for all j=0,1,....,2p and j=0,1,...., p; $\vec{\alpha}^{1} = (\alpha_0,\alpha_1,....\alpha_p)$ and $\vec{t}^{1} = (t_0, t_1,...,t_p)$.

For notation convenience, let

$$T_{p+1,d} = \begin{bmatrix} S_0 & \cdots & S_{d-1} & t_0 & S_{d+1} & \cdots & S_p \\ S_1 & \cdots & S_{d+1-1} & t_1 & S_{d+2} & \cdots & S_{p+1} \\ . & \cdots & . & . & . & \cdots & . \\ . & \cdots & . & . & . & \cdots & . \\ . & \cdots & . & . & . & \cdots & . \\ S_p & \cdots & S_{p+d-1} & t_p & S_{p+d+1} & \cdots & S_{2p} \end{bmatrix},$$

$$U_{p+1,d}\left(\frac{X_i-x}{h}\right) = \begin{bmatrix} S_0 & S_1 & \cdots & S_{d-1} & 1 & S_{d+1} & \cdots & S_p \\ S_1 & S_2 & \cdots & S_{d+1-1} & (X_i-x)/h & S_{d+2} & \cdots & S_{p+1} \\ . & . & \cdots & . & . & . & \cdots & . \\ . & . & \cdots & . & . & . & \cdots & . \\ S_p & S_{p+1} & \cdots & S_{p+d-1} & ((X_i-x)/h)^p & S_{P+d+1} & \cdots & S_{2p} \end{bmatrix},$$

for all $d = 1,2,...,p \geq 1$. From (1.2) and the features of the MQF, one will construct a KRE such that it does not have the boundary effects. In practice, the new KRE is easily constructed by minimizing quadratic form as above. Now, one minimize (1.2) with respect to $\vec{\alpha}$, denoting the solution of $\vec{\alpha}$ by $\hat{\vec{\alpha}}$

Further to calculate at (1.2) and use the Cramer's Rule, the estimator of $^2\alpha$, d = 1,2,...,p can be expressed as

$$\hat{\alpha}_d = \hat{g}^{(d)}_{p+1}(x) = \frac{|T_{p+1,d}|}{|A|} = \hat{g}^{(d)}_{p+1}(x;h_1)$$

$$= \frac{1}{nh^{d+1}} \sum_{i=1}^{n} \frac{\left|U_{p+1,d}(\frac{X_i - x}{h})\right|}{|A|} K\left\{ \frac{X_i - x}{h} \right\} \frac{Y_i}{f(X_i)} \quad , \tag{1.5}$$

where $T_{p+1,d}$, A and $U_{p+1,d}(\cdot)$ are defined as above, for $d = 1,2,...,p \geq 1$. Here $|\cdot|$ is the definition of determinant.

Let $(d!)\,\hat{g}^{(d)}_{p+1}(x;h) = \tilde{m}^{(d)}_{p+1}(x;h)$. Here one uses the estimator of $\tilde{m}^{(d)}_{p+1}$(x; h) to estimate $m^{(d)}(x)$, for all $d = 1,2,...,p \geq 1$. From (1.5), we known that it does have a nice property which is given by

$$\frac{1}{h} \int K\left\{ \frac{u-x}{h} \right\} \left|U_{p+1,d}(\frac{u-x}{h})\right| (\frac{u-x}{h})^q \, du = 0 \quad , \tag{1.6}$$

for all $q = 0,1,2,...,d\text{-}1,d+1,..., p \geq 1$. We know that the result of (1.6) as above is true by the properties of linear algebra (Anton, 1984).

The procedure of estimation of $m^{(d)}$ by $\tilde{m}^{(d)}_{p+1}(x;h)$ can be further improved the KRE of Mack and Müller (1989), because it does not have the problem of boundary effects and be proved in the next section. The bias of $\tilde{m}^{(d)}_{p+1}$(x; h) can be also reduced and it is also a weighting KRE. This paper is organized as follows: In section 2, one investigates the asymptotic results of $\tilde{m}^{(d)}_{p+1}$(x; h) and gives the explicit formula for mean integrated square error (MISE) (or mean square error, MSE) and optimal bandwidth. In section 3, one gives the proofs of the asymptotic results.

2. Asymptotic results

In this section, we investigate the asymptotic results of $\tilde{m}^{(d)}_{p+1}$(x; h). Before we state the asymptotic results we will give the following assumptions:

(A1) Let $(X_1, Y_1)_2,....,(X_n, Y_n)$, as before, be independent and identically distributed with the joint density function $f(x,y)$, m(x) and $f(x)$ are bounded on its domain, and continuous at the point x, $m^{(q)}(x)$ and $f^{(q)}(x)$ exists, for $q \geq p+1$ and $p \geq 1$. Let $v^2(x) = E[Y^2 \mid X = x]$ is bounded on its domain and continuous at the point x.

(A2) K and W are kernel functions and it satisfies.

(i) $\limsup_{|z|\to\infty} \left| z^{k+p+4} K(z) \right| < \infty$, (ii) $\limsup_{|z|\to\infty} \left| z^{k+p+1} W(z) \right| < \infty$,

(iii) $\limsup_{z\to-\infty} \left| z^{k+4} K(z) \right| < \infty$, (iv) $\limsup_{z\to\infty} \left| z^{k+4} K(z) \right| < \infty$ and

(v) $\limsup_{|z|\to\infty} \left| z^{k+2} W(z) \right| < \infty$, for all $k \geq 1$ and $p \geq 1$.

For notation convenience, let

$$D_{p+1} = \begin{bmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_p \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{p+1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ \mu_p & \mu_{p+1} & \mu_{p+2} & \cdots & \mu_{2p} \end{bmatrix},$$

$$J_{p+1,d} = \begin{bmatrix} \mu_0 & \cdots & \mu_{d-1} & \mu_{P+1} & \mu_{d+1} & \cdots & \mu_p \\ \mu_1 & \cdots & \mu_{d+1-1} & \mu_{p+2} & \mu_{d+1+1} & \cdots & \mu_{p+1} \\ . & \cdots & . & . & . & \cdots & . \\ . & \cdots & . & . & . & \cdots & . \\ . & \cdots & . & . & . & \cdots & . \\ \mu_p & \cdots & \mu_{p+d-1} & \mu_{2p+1} & \mu_{p+d+1} & \cdots & \mu_{2p} \end{bmatrix},$$

$$N_{p+1,d} = \begin{bmatrix} \mu_0 & \cdots & \mu_{d-1} & 1 & \mu_{d+1} & \cdots & \mu_p \\ \mu_1 & \cdots & \mu_{d+1-1} & u & \mu_{d+1+1} & \cdots & \mu_{p+1} \\ . & \cdots & . & . & . & \cdots & . \\ . & \cdots & . & . & . & \cdots & . \\ . & \cdots & . & . & . & \cdots & . \\ \mu_p & \cdots & \mu_{p+d-1} & u^p & \mu_{p+d+1} & \cdots & \mu_{2p} \end{bmatrix},$$

for all $d = 1,2,\ldots,p \geq 1$.

Let us now give the asymptotic results of the estimator $\tilde{m}_{p+1}^{(d)}$(x; h). In the following theorem 2.1, one gives the asymptotic bias and variance of the estimator at the point $x \in (-\infty, \infty)$.

Theorem 2.1. Assume that conditions (A1) and (A2) — (i) are satisfied. And suppose that $h \to 0$ and $nh \to \infty$, as $n \to \infty$, then the asymptotic bias and the asymptotic variance of the estimator $\tilde{m}_{p+1}^{(d)}$(x; h) at the point $x \in (-\infty, \infty)$ can be given by

$$Bias\left[\tilde{m}_{p+1}^{(d)}(x;h)\right] = \frac{d!}{(p+1)!} h^{p+1-d} \left[\frac{\left|J_{p+1,d}\right|}{\left|D_{p+1}\right|}\right] m^{(p+1)}(x) + o(h^{p+1-d}) , \tag{2.1}$$

$$Var\left[\tilde{m}_{p+1}^{(d)}(x;h)\right] = \frac{(d!)^2}{nh^{2d+1}} \frac{v^2(x)}{f(x)} \frac{\int_{-\infty}^{\infty} K^2(t)\left[\left|N_{p+1,d}\right|\right]^2 dt}{\left[\left|D_{p+1}\right|\right]^2} + o\left(\frac{1}{nh^{2d+1}}\right) , \tag{2.2}$$

where $\mu_j = \int_{-\infty}^{\infty} v^j K(v) dv$, for $j = 0,1,\ldots,2p+1$. (2.3)

By the result of Theorem 2.1 as above, we do not need to require that the kernel function has the mean of zero and integrates to one here. If $\mu_j = 0$, j is odd, the bias of $\tilde{m}_{p+1}^{(d)}(x;h)$ can be reduced. The details can also refer to Gasser and Müller (1979). Therefore, the proposed method can help us to construct a high-order KRE.

If $m^{(p+1)}(x)$ is a continuous function and square integral on $(-\infty, \infty)$, we will have the following theorem about the MISE of the estimator $\tilde{m}_{p+1}^{(d)}(x;h)$ at the interior points.

Theorem2.2. Under the assumptions of Theorem 2.1 and assume that $m^{(d)}(x)$ is continuous, then the MISE of $\tilde{m}_{p+1}^{(d)}(x;h)$ is

$$MISE(n;h) = h^{2p+2-2d}B + n^{-1}h^{-2d-1}V + o_p\left(h^{2p+2-2d} + \frac{1}{nh^{2d+1}}\right) , \tag{2.4}$$

where $B = \frac{(d!)^2}{((p+1)!)^2} \left[\frac{\left|J_{p+1,d}\right|}{\left|D_{p+1}\right|}\right]^2 \int_{-\infty}^{\infty} \left[m^{(p+1)}(x)\right]^2 dx$, (2.5)

and $V = (d!)^2 \int_{-\infty}^{\infty} \frac{v^2(x)}{f(x)} dx \left(\int_{-\infty}^{\infty} K^2(t) \left[\left|N_{p+1,d}\right|\right]^2 dt \Big/ \left[\left|D_{p+1}\right|\right]^2\right)$, (2.6) with μ_j ($j = 0,1,...,2p+1$) is defined as above and $d = 1,2,...,p \geq 1$.

From the result of (2.4), we can get the optimal bandwidth h for minimizing of MISE, denoted by h(opt) and given by

$$h(opt) = M \cdot n^{-1/(2p+3)}(1+o(1)), \tag{2.7}$$

where $$M = \left(\frac{2d+1}{2p+2-2d}\right)^{\frac{1}{2p+3}} B^{-1/(2p+3)} \cdot V^{1/(2p+3)}, \tag{2.8}$$

with B and V defined at (2.5) and (2.6), respectively.

Remark 1: The optimal bandwidth h depends on the kernel function K and the function $m^{(p+1)}(x)$. In practice, we need to choose of the kernel function K and the adaptive estimator of $m^{(p+1)}(x)$. From Theorem 2.1, we can take the optimal (or Epanechnikov) kernel function, for example, $K(u) = 0.75$ $(1-u^2)$, for all $u \in [-1, 1]$. To choose the smoother parameter h can refer to, for examples, Silverman (1986) and Härdle (1991).

Next, one considers the boundary problem of the estimator $\tilde{m}_{p+1}^{(d)}(x; h)$. At this time, the domain of $m(x)$ is $[a,b]$. Without loss of generality, one only considers $p = 1$ and $d = 1$ case. In fact, we only need to consider the left and right boundary points here. One first investigates the behavior of $\tilde{m}_2^{(1)}(x;h)$ at left-boundary points. Set $x = a+\eta h$ with $1 > \eta \geq 0$. In the following theorem we give the asymptotic bias and the asymptotic variance at left-boundary points.

Theorem 2.3. Assume that condition (A1) and (A2)-(iii) are satisfied. Furthermore, suppose that $m^{(2)}(x)$ is right continuous at the point a, $h \to 0$ and $nh \to \infty$. Then, the asymptotic bias and the asymptotic variance of $\tilde{m}_2^{(1)}(x;h)$ at the boundary point $x = a+\eta h$ can be given as follows:

$$Bias\left[\tilde{m}_2^{(1)}(a^+;h)\right] = \frac{1}{2}h\left[\frac{\mu_{0,\eta}\mu_{3,\eta} - \mu_{1,\eta}\mu_{2,\eta}}{\mu_{0,\eta}\mu_{2,\eta} - \mu_{1,\eta}^2}\right] m^{(2)}(a^+) + o(h), \tag{2.9}$$

$$Var\left[\tilde{m}_2^{(1)}(a^+;h)\right] = \frac{1}{nh^3}\frac{v^2(a^+)}{f(a^+)}\frac{\int_{-\infty}^{\eta} K^2(t)\left[\mu_{0,\eta}t - \mu_{1,\eta}\right]^2 dt}{\left[\mu_{0,\eta}\mu_{2,\eta} - \mu_{1,\eta}^2\right]^2} + o\left(\frac{1}{nh^3}\right), \tag{2.10}$$

where $\mu_{j,\eta} = \int_{-\infty}^{\eta} v^j K(v)dv$, for $j = 0,1,2,3$. (2.11)

Secondly, we investigate the behavior of $\tilde{m}_2^{(1)}(x;h)$ at right-boundary points. Given $x = a+\eta h$ with $1 > \eta \geq 0$, applying the way of Theorem 2.3, we have the following theorem.

Theorem 2.4. Assume that conditions (A1) and (A2)-(iv) are satisfied. Furthermore, suppose that $m^{(2)}(x)$ is left continuous at the point b, $h \to 0$ and $nh \to \infty$. Then, the asymptotic bias and the asymptotic variance of $\tilde{m}_2^{(1)}(x;h)$ at the boundary point $x = b-\eta h$ can be represented by

$$Bias\left[\tilde{m}_2^{(1)}(b^-;h)\right] = \frac{1}{2}h\left[\frac{\mu_{0,\eta_1}\mu_{3,\eta_1} - \mu_{1,\eta_1}\mu_{2,\eta_1}}{\mu_{0,\eta_1}\mu_{2,\eta_1} - \mu_{1,\eta_1}^2}\right]m^{(2)}(b^-) + o(h) \ , \tag{2.12}$$

$$Var\left[\tilde{m}_2^{(1)}(b^-;h)\right] = \frac{1}{nh^3}\frac{v^2(b^-)}{f(b^-)}\frac{\int_{-\eta}^{\infty} K^2(t)\left[t\mu_{0,\eta_1} - \mu_{1,\eta_1}\right]^2 dt}{\left[\mu_{0,\eta_1}\mu_{2,\eta_1} - \mu_{1,\eta_1}^2\right]^2} + o\left(\frac{1}{nh^3}\right) , \tag{2.13}$$

where $\mu_{j,\eta_1} = \int_{-\eta}^{\infty} v^j K(v)dv$, for $j = 0,1,2,3$. (2.14)

The proof of Theorem 2.4 follows that of Theorem 2.3.

Two immediate corollaries from the Theorem 2.3 and Theorem 2.4 can be stated as follows:

Corollary 1. Under the assumptions of Theorem 2.3, we have the mean squared error (MSE) of $\tilde{m}_2^{(1)}(x;h)$ at the left-boundary points which is given by

$$MSE(n;h) = h^2B^* + n^{-1}h^{-3}V^* + o_p(h^2 + n^{-1}h^{-3}) \ , \tag{2.15}$$

where $B^* = \frac{1}{4}\left[\frac{\mu_{0,\eta}\mu_{3,\eta} - \mu_{1,\eta}\mu_{2,\eta}}{\mu_{0,\eta}\mu_{2,\eta} - \mu_{1,\eta}^2}\right]^2 (m^{(2)}(a^+))^2$,

and $V^* = \left\{\int_{-\infty}^{\eta} K^2(t)\left[\mu_{0,\eta}t - \mu_{1,\eta}\right]^2 dt \Big/ \left[\mu_{0,\eta}\mu_{2,\eta} - \mu_{1,\eta}^2\right]^2\right\}\frac{v^2(a^+)}{f(a^+)}$,

with $\mu_{j,\eta}$ is defined as above. (j=0,1,2,3)

Corollary 2. Under the assumptions of Theorem 2.4, we have the MSE of $\tilde{m}_{2}^{(1)}(x;h)$ at the right-boundary points which is given by

$$MSE(n;h) = h^2 B^{**} + n^{-1}h^{-3}V^{**} + o_p(h^2 + n^{-1}h^{-3}) , \tag{2.16}$$

where $B^{**} = \frac{1}{4}\left[\frac{\mu_{0,\eta_1}\mu_{3,\eta_1} - \mu_{1,\eta_1}\mu_{2,\eta_1}}{\mu_{0,\eta_1}\mu_{2,\eta_1} - \mu_{1,\eta_1}^2}\right]^2 (m^{(2)}(b^-))^2$,

and $V^{*} = \left\{\int_{-\eta}^{\infty} K^2(t)\left[\mu_{0,\eta_1}t - \mu_{1,\eta_1}\right]^2 dt \Big/ \left[\mu_{0,\eta_1}\mu_{2,\eta_1} - \mu_{1,\eta_1}^2\right]^2\right\}\frac{v^2(b^-)}{f(b^-)}$,

with $\mu_{j,\eta 1}$ is defined as above. (j=0,1,2,3)

From Theorem 2.1 and Theorem 2.3-4, we can obtain the following corollary:

Corollary 3. Assume that condition (A1) and (A2) are satisfied. And suppose that $h \to 0$ and $nh^{2d+1} \to \infty$, as $n \to \infty$, then the asymptotic bias and the asymptotic variance of $\tilde{m}_{p+1}^{(d)}(x;\, h)$ at the point $x \in [a,b]$ can be given by

$$Bias\left[\tilde{m}_{p+1}^{(d)}(x;h)\right] = O\left(h^{p+1-d}\right) , \tag{2.17}$$

$$Var\left[\tilde{m}_{p+1}^{(d)}(x;h)\right] = O\left(\frac{1}{nh^{2d+1}}\right) , \tag{2.18}$$

Remark 2: It follows from Theorem 2.3-4 that the estimator $\tilde{m}_{p+1}^{(d)}(x;\, h)$ does not have boundary effects. Its rate of convergence is indeed improved the KRE of Mack and Müller (1989). Applying the minimizing quadratic form method, it does not require modifications on the boundary regions. The bias of proposed estimator is achieved $O(h^{p+1-d})$ and the optimal bandwidth of h is achieved $O(n^{\frac{-1}{2p+3}})$.

Remark 3: The estimation of regression curves can also refer to the method of local linear smoother by Wei and Chu (1994) and Fan and Gijbels (1992) for estimating $m(x)$.

Remark 4: The estimator of $\tilde{m}_{p+1}^{(d)}(x;\, h)$ is depends on the density function f(x).

As f(x) is unknown, we need further to estimate $\tilde{m}_{p+1}^{(d)}(x;\, h)$ such that it is a practical KRE. The details can refer to the paper of Mack and Müller (1989).

From Theorem 2.1, we can obtain the asymptotic distribution of $\tilde{m}^{(d)}_{p+1}(x; h)$. Due to the asymptotic distribution can help us to construct the confidence interval. Therefore, we have the following theorem.

Theorem2.5. Under the assumptions of Theorem 2.1, and that $nh^{2p+3} \to r^2 > 0$ as $n \to \infty$, we have

$$\sqrt{nh^{2d+1}}\left[\tilde{m}^{(d)}_{p+1}(x;h) - m^{(d)}(x)\right] \to N(\mu(x), \sigma^2(x)) \quad , \tag{2.19}$$

in distribution as $n \to \infty$, where

$$\mu(x) = \frac{r \cdot (d!)}{(p+1)!}\left[\frac{|J_{p+1,d}|}{|D_{p+1}|}\right] m^{(p+1)}(x) ,$$

and

$$\sigma^2(x) = (d!)^2 \frac{v^2(x)}{f(x)} \frac{\int_{-\infty}^{\infty} K^2(t)\left[|N_{p+1,d}|\right]^2 dt}{\left[|D_{p+1}|\right]^2} ,$$

for $d = 1,2,\dots,p \geq 1$.

3. Proofs: In this section our main purpose is to prove Theorem 2.1-5. First, one will introduce four auxiliary lemmas which is given as follows:

Lemma3.1. Assume that $m(\cdot)$ and $m^{(2)}(\cdot)$ are bounded on $[a, b]$, and right continuous at the point a. Set $x = a+\eta h$ with $1 > \eta \geq 0$. Furthermore, suppose that $h \to 0$, $nh^3 \to \infty$ and the condition (A2)-(iii) is satisfied. Then

$$\begin{aligned} &\int_a^b \frac{1}{h} K\left\{\frac{u-x}{h}\right\}\left(\frac{u-x}{h}\right)^m g(u)du \\ &= \frac{1}{2} h^2 m^{(2)}(a^+) \int_{-\infty}^{\eta} K(v) v^{m+2} dv (1 + o_p(1)) \\ &= \frac{1}{2} h^2 m^{(2)}(a^+) \cdot \mu_{m+2,\eta}(1 + o_p(1)) \quad , \end{aligned} \tag{3.1}$$

where $g(u) = m(u)-m(x)-m^{(1)}(x)(u-x)$ and

$$\mu_{m+2,\eta} = \int_{-\infty}^{\eta} K(v) v^{m+2} dv \quad ,$$

for all m = 0,1,2....

Proof: The proof follows the Lemma 1 of Fan and Gijbels (1992).

Lemma3.2. Assume that $m^{(2)}(\cdot)$ and $L(\cdot)$ are bounded on $[a, b]$ and right continuous at the point a. Let $x = a+\eta h$ with $1 > \eta \geq 0$. Furthermore, suppose that (A2)-(v) is satisfied and $h \rightarrow 0$. Then

$$\int_a^b \frac{1}{h} W\left\{ \frac{u-x}{h} \right\}\left(\frac{u-x}{h} \right)^m L(u)du \tag{3.2}$$
$$= L(a^+)\int_{-\infty}^{\eta} W(v)v^m dv(1+o_p(1))$$
$$= L(a^+)\tilde{\mu}_{m,\eta}(1+o_p(1)) \ ,$$

where $\tilde{\mu}_{m,\eta} = \int_{-\infty}^{\eta} W(v)v^m dv$.

Proof: The proof follows that of Lemma 3.1.

Lemma 3.3. Assume that $m^{(p+1)}(\cdot)$ are bounded and continuous at the point $x \in (-\infty, \infty)$ for all $p \geq 1$. Furthermore, suppose that $h \rightarrow 0$ and the condition (A2)-(i) is satisfied. Then

$$\int_{-\infty}^{\infty} \frac{1}{h} K\left\{ \frac{u-x}{h} \right\}\left(\frac{u-x}{h} \right)^m g(u)du \tag{3.3}$$
$$= \frac{1}{(p+1)!} h^{p+1} m^{(p+1)}(x)\int_{-\infty}^{\infty} K(v)v^{m+p+1}dv(1+o_p(1))$$
$$= \frac{1}{(p+1)!} h^{p+1} m^{(p+1)}(x)\mu_{m+p+1}(1+o_p(1)) \ ,$$

where $\mu_{m+p+1} = \int_{-\infty}^{\infty} K(v)v^{m+p+1}dv$ and

$$g(u) = m(u) - m(x) - (u-x)m^{(1)}(x) - \cdots - \frac{1}{p!}(u-x)^p m^{(p)}(x) \ .$$

Proof: The proof also follows that of Lemma 3.1.

Lemma 3.4. Assume that $m^{(p+1)}(\cdot)$ and $L(\cdot)$ are continuous at the point $x \in (-\infty, \infty)$. Furthermore, suppose that $h \rightarrow 0$ and the condition (A2)-(ii) is fulfilled. Then

$$\int_{-\infty}^{\infty} \frac{1}{h} W\left\{ \frac{u-x}{h} \right\} \left(\frac{u-x}{h} \right)^m L(u)du \tag{3.4}$$
$$= L(x)\int_{-\infty}^{\infty} W(v)v^m dv(1+o_p(1))$$
$$= L(x)\tilde{\mu}_m (1+o_p(1)) \ ,$$

where $\tilde{\mu}_m = \int_{-\infty}^{\infty} W(v)v^m dv$.

Proof: The proof is similar to that of Lemma 3.3.

For convenience, let $B_1 = S_0S_2\text{-}S_1{}^2$, and S_j is defined at (1.5), for $j = 0,1,2,3$.

Proof of Theorem2.3. Set $x = a+\eta h$ in Theorem 2.3. From the evaluation of the bias of the estimator $\tilde{m}_2^{(1)}(x;h)$ which is given by

$$E\left[\tilde{m}_2^{(1)}(x;h)\right]$$
$$= \frac{1}{h^2}\int_a^b K\left\{ \frac{u-x}{h} \right\}\left[S_o\left(\frac{u-x}{h} \right) - S_1 \right] m(u)\frac{f(u)}{f(u)}du / B_1$$
$$= \frac{1}{h^2}\int_a^b K\left\{ \frac{u-x}{h} \right\}\left[S_o\left(\frac{u-x}{h} \right) - S_1 \right] m(u)du / B_1$$
$$+\frac{1}{h^2}\int_a^b K\left\{ \frac{u-x}{h} \right\}\left[S_o\left(\frac{u-x}{h} \right) - S_1 \right] (u-x)m^{(1)}(x)du / B_1$$
$$+\frac{1}{h^2}\int_a^b K\left\{ \frac{u-x}{h} \right\}\left[S_o\left(\frac{u-x}{h} \right) - S_1 \right]\left[m(u)-m(x)-(u-x)m^{(1)}(x) \right] du / B_1$$

$$= m^{(1)}(x) + \frac{1}{h^2}\int_a^b K\left\{ \frac{u-x}{h} \right\}\left[S_o\left(\frac{u-x}{h} \right) - S_1 \right] (u-x)g(u)du / B_1 \tag{3.5}$$
$$(by \quad \int_a^b K\left\{ \frac{u-x}{h} \right\}\left[S_o\left(\frac{u-x}{h} \right) - S_1 \right] du = 0)$$
$$= m^{(1)}(x) + \frac{1}{h^2}\left[S_o\int_a^b \frac{1}{h} K\left\{ \frac{u-x}{h} \right\}\frac{(u-x)}{h} g(u)du - \frac{1}{h^2} S_1 \int_a^b K\left\{ \frac{u-x}{h} \right\} g(u)du \right] / B_1 \ ,$$

where $g(u) = m(u)\text{-}m(x)\text{-}(u\text{-}x)m^{(1)}(x)$.

For convenience, let $M(\eta) = (\mu_{0,\eta}\mu_{2,\eta} - \mu_{1,\eta}^2)$. Applying (3.5), Lemma 3.1 and Lemma 3.2, we have

$$E\left[\tilde{m}_2^{(1)}(a^+;h)\right]$$
$$= m^{(1)}(a^+)+\frac{1}{2}h\left[\mu_{0,\eta}\mu_{3,\eta}-\mu_{1,\eta}\mu_{2,\eta}\right]m^{(2)}(a^+)(1+o_p(1))/(M(\eta)(1+o_p(1))) \ ,$$

apply $x = a+\eta h \rightarrow a^+$, as $h \rightarrow 0$; thus, the proof of the asymptotic bias is proved.

Next, we calculate the asymptotic variance of $\tilde{m}_2^{(1)}(x;h)$. The estimator $\tilde{m}_2^{(1)}(x;h)$ is rewritten as below:

$$\tilde{m}_2^{(1)}(x;h) = \sum_{j=1}^{n}\frac{1}{nh^2}K\left\{\frac{X_i-x}{h}\right\}\left[S_o\left(\frac{X_i-x}{h}\right)-S_1\right]\frac{Y_j}{f(X_j)}\Big/B_1 \tag{3.6}$$

Applying the evaluation of the variance, we have

$$Var\left[\tilde{m}_2^{(1)}(x;h)\right]$$
$$=\frac{1}{nh^4}\int_a^b\left\{K\left\{\frac{u-x}{h}\right\}\left[S_0\left(\frac{u-x}{h}\right)-S_1\right]\right\}^2\nu^2(u)\frac{f(u)}{f^2(u)}du/B_1^2$$
$$-n\left(\left\{\frac{1}{nh^2}\int_a^b K\left\{\frac{u-x}{h}\right\}\left[S_0\left(\frac{u-x}{h}\right)-S_1\right]m(u)du/B_1\right\}\right)^2$$
$$=\frac{1}{nh^3}\int_a^b\frac{1}{h}\left\{K\left\{\frac{u-x}{h}\right\}\left[S_0\left(\frac{u-x}{h}\right)-S_1\right]\right\}^2\frac{\nu^2(u)}{f(u)}du/B_1^2+o\left(\frac{1}{nh^3}\right)$$

(by the calculation of the bias, Lemma 3.1 and Lemma 3.2.)

$$=\frac{\nu^2(x)}{nh^3f(x)}\int_{-\infty}^{\eta}\left[\mu_{0,\eta}tK(t)-\mu_{1,\eta}K(t)\right]^2dt\Big/\left[\mu_{0,\eta}\mu_{2,\eta}-\mu_{1,\eta}^2\right]^2+o_p\left(\frac{1}{nh^3}\right). \tag{3.5}$$

Applying $x = a+\eta h \rightarrow a^+$, as $h \rightarrow 0$, the proof of the asymptotic variance is proved. Hence the proof of the Theorem 2.3 is completed as above.

Proof of Theorem 2.4. The proof follows the same techniques of Theorem 2.3, at this time, x is replaced by b-ηh.

Proof of Theorem 2.1. The proof follows the same techniques of Theorem 2.3, at this time, Lemma 3.1 and Lemma 3.2 are replaced by Lemma 3.3 and Lemma 3.4 for $x \in (-\infty, \infty)$. Here we are also used to the good properties at (1.6).

Proof of Theorem 2.2. The proof follows from the Theorem 2.1 and the calculation of MISE.

Proof of Theorem 2.5. The proof follows from the Theorem 2.1 and the book of Prakasa Rao (1983).

4. Conclusion

In this paper researcher uses the good property of minimizing quadratic form (or least squares method) to construct a general KRE. We know that the asymptotic bias and variance of the proposed estimator are very easy to calculate in this paper for estimating $m^{(d)}(d \geq 1)$. By Theorem 2.3-4, the proposed estimator can help us to solve the problem of boundary effects underlying the bounded domain of the regression curves. And the kernel function cannot give any required. The proposed estimator can be improved the KRE of Mack and Müller (1989). The rate of convergence of $\tilde{m}^{(d)}_{p+1}(x; h)$ is achieved $O(n^{-(2p+2-2d)/(2p+3)})$, for all $x \in [a, b]$(or real line) and d = 1,2, ...,p ≥ 1. In practice, the function of f(x) is unknown, the proposed estimator will be further to estimate in the future.

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References

Anton, H. (1984). *Elementary linear algebra.* New York: John Wiley and Sons.

Fan, J. and Gijbels, I. (1992). Variable bandwidth and local linear regression smoothers. *Annals Statistic* 20, 2008-2036.

Gasser, T. & Müller, H.G. (1979). Kernel estimation of regression functions. *Smoothing Techniques for Curve Estimation.* Eds. Gasser and Rosenblatt, Springer-Verlag, Heidelberg, 23-68.

Gasser, T. & Müller, H.G. (1984). Estimating regression functions and their derivatives by the kernel method. *Scandinavian Journal of Statistics,* 11, 171-185.

Härdle, W. (1991). *Smoothing Techniques With Implement in S.* New York: Springer-Verlag.

Müller, H.G. (1993). On the boundary kernel method for nonparametric curve estimation near endpoints. *Scandinavian Journal of Statistics* 20, 313-328.

Mack, Y. P. & Müller, H.G. (1989). Derivative estimation in nonparametric regression with random predictor variable. *Sankhy: The Indian Journal of Statistics* 51, Series A, 59-72.

Prakasa Rao, B.I.S. (1983). *Nonparametic Functional Estimation.* New York: Academic Press.

Ruppert, D. & Wand, M.P. (1994). Multivariate locally weighted least squares regression. *Annals Statistic* 22, 1346-1370.

Simonoff, J.S. (1996). *Smoothing Methods in Statistics.* New York: Springer-Verlag.

Silverman, B.W. (1986). *Density Estimation for Statistics and Data analysis.* London: Chapman and Hall.

Wand, M.P. & Jones, M.C. (1995). *Kernel Smoothing.* London: Chapman and Hall.

Wei, C.Z. & Chu, C.K. (1994). A Regression Point of View Toward Desity Estimation. *Journal of Nonparametric Statistics* 4, 191-201.

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迴歸曲線之導數的核估評估

洪萬吉

摘　要

在無母迴歸估計中，迴歸曲線之導數的估計已被研究。當密度函數 f(x)之定義域為有界時， Mack 與 Müller（1989）所提估計迴歸曲線之導數的核迴歸估計量也遭受所謂的邊界效果問題。為了改善上述之邊界效果與核估計量之偏差減小, 在本文研究者依最小二次方程式來建構一個新的核迴歸估計量。對所提之核迴歸估計量將提供其近似偏差與近似變異數之簡潔式及其一些性質。除此之外，所提核迴歸估計量將不會產生邊界效果且可改善 Mack 與 Müller（1989）之核迴歸估計量之缺失，其收斂速率達 $O(n^{-\frac{p+1-d}{2p+3}})$ ， $d = 1,2,\ldots,p \geq 1$ 。

關鍵詞：核迴歸估計量，邊界效果，導數，整合均方差，偏差減小

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