費玻納西數列

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摘 要

吾人在此論文中，欲考慮數列 $u_1=\alpha$ ，$u_2=\beta$ ，$u_{n+1}=p\,u_{n-1}+q\,u_n$ （其中n爲整數，α與β爲任意不全爲0之二複數，p與q爲正實數）之若干性質。

最深值注意之結果爲：

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\frac{q+\sqrt{q^2+4p}}{2}\qquad\left(若\ \beta-\frac{q-\sqrt{q^2+4p}}{2}\alpha\neq0\right)$$

$$=\frac{q-\sqrt{q^2+4p}}{2}\qquad\left(若\ \beta-\frac{q-\sqrt{q^2+4p}}{2}\alpha=0\right)$$

$$\lim_{n\to-\infty}\frac{u_n}{u_{n+1}}=\frac{2}{q-\sqrt{q^2+4p}}\qquad\left(若\ \beta-\frac{q+\sqrt{q^2+4p}}{2}\alpha\neq0\right)$$

$$=\frac{2}{q+\sqrt{q^2+4p}}\qquad\left(若\ \beta-\frac{q+\sqrt{q^2+4p}}{2}\alpha=0\right)$$

"$\frac{\beta}{\alpha} = (q-\sqrt{q^2+4p})/2 \not\Rightarrow (u_n)_{n\in Z}$ is alternating."
It is clear that

"$(u_n)_{n\in Z}$ is alternating $\Rightarrow (u_n)_{n\in N}$ is alternating."

Now, we want to show that

"$(u_n)_{n\in N}$ is alternating $\Rightarrow \beta/\alpha = (q-\sqrt{q^2+4p})/2$."

Suppose that $\beta/\alpha \neq (q-\sqrt{q^2+4p})/2$, then by theorem 1, $\frac{u_{n+1}}{u_n} \longrightarrow \frac{q+\sqrt{q^2+4p}}{2} > 0$ as $n \to \infty$, which implies that $(u_n)_{n\in N}$ is not alternating. By above arguments, we obtain

THEOREM 4 : $\frac{\beta}{\alpha} = \frac{q-\sqrt{q^2+4p}}{2} \Leftrightarrow (u_n)_{n\in Z}$ is alternating $\Leftrightarrow (u_n)_{n\in N}$ is alternating.

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Moreover, if $n \leq 0$, take $V^0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then

$UV^0 = U = \begin{pmatrix} u_2 & u_1 \\ u_1 & u_0 \end{pmatrix}$, hence (***) is true for $n=0$.

Now suppose that (***) is true for some negative integer k, that is, $UV^k = \begin{pmatrix} u_{k+2} & u_{k+1} \\ u_{k+1} & u_k \end{pmatrix}$, then

$$UV^{k-1} = UV^kV^{-1} = \begin{pmatrix} u_{k+2} & u_{k+1} \\ u_{k+1} & u_k \end{pmatrix} \begin{pmatrix} 0 & 1/p \\ 1 & -1/p \end{pmatrix} = \begin{pmatrix} u_{k+1} & (u_{k+2} - q u_{k+1})/p \\ u_k & (u_{k+1} - q u_k)/p \end{pmatrix}$$

$$= \begin{pmatrix} u_{k+1} & u_k \\ u_k & u_{k-1} \end{pmatrix}$$, hence (***) is true for all $n \in Z^-$.

Therefore, we have the following

LEMMA 2 : If $U = \begin{pmatrix} u_2 & u_1 \\ u_1 & u_0 \end{pmatrix}$, $V = \begin{pmatrix} q & 1 \\ p & 0 \end{pmatrix}$, then

$$UV^n = \begin{pmatrix} u_{n+2} & u_{n+1} \\ u_{n+1} & u_n \end{pmatrix} \quad \text{for all } n \in Z.$$

Now since

$$|UV^n| = \begin{vmatrix} u_{n+2} & u_{n+1} \\ u_{n+1} & u_n \end{vmatrix} = u_{n+2}u_n - u_{n+1}^2$$

and

$$|U| = \begin{vmatrix} u_2 & u_1 \\ u_1 & u_0 \end{vmatrix} = u_2u_0 - u_1^2 = (\beta^2 - q\alpha\beta - p\alpha^2)/p,$$

$|V^n| = |V|^n = (-p)^n$, hence we have

THEOREM 3 : For $n \in Z$, $u_{n+2}u_n - (u_{n+1})^2 = (-1)^n p^{n-1}(\beta^2 - q\alpha\beta - p\alpha^2)$

Now if $\beta/\alpha = (q - \sqrt{q^2+4p})/2$, that is, $u_2 = su_1$, then $\beta^2 - q\alpha\beta - p\alpha^2 = 0$, hence by theorem 3, $(u_{n+1})^2 = u_{n+2}u_n$, thus by lemma 1 of this part, $u_k \neq 0$ for all $k \in Z$. Hence $(u_n)_{n \in Z}$ is a geometric progression with ration $\beta/\alpha = (q - \sqrt{q^2+4p})/2 < 0$, thus, $(u_n)_{n \in Z}$ is alternating. By now, we have

$$u_{n-1} = \frac{rs^{n-2}-r^{n-2}s}{r-s}\ u_1 + \frac{r^{n-2}-s^{n-2}}{r-s}\ u_2 \ \cdots\cdots\cdots\cdots(**)$$

Now if $ru_1-u_2\neq 0$, thon $|r/s|^{n-1}\to 0$ as $n\to-\infty$, hence

$$\frac{u_n}{u_{n+1}} = \frac{(r-s(r/s)^{n-1})u_1 + ((r/s)^{n-1}-1)u_2}{(rs-rs(r/s)^{n-1})u_1 + ((r/s)^{n-1}r-s)u_2} \longrightarrow \frac{ru_1-u_2}{rsu_1-su_2}$$

$$= \frac{1}{s} = \frac{2}{q-\sqrt{q^2+4p}} \text{ as } n\longrightarrow-\infty .$$

If $ru_1-u_2=0$, then by (**), we know that

$$\frac{u_n}{u_{n+1}} = \frac{(r^{n-1}-s^{n-1})u_2 + (rs^{n-1}-r^{n-1}s)u_1}{(r^n-s^n)u_2 + (rs^n-r^ns)u_1} = \frac{(r^n-r^{n-1}s)u_1}{(r^{n+1}-r^ns)u_1}$$

$$= \frac{1}{r} = \frac{2}{q+\sqrt{q^2+4p}}$$

Hence

$$\frac{u_n}{u_{n+1}} \longrightarrow \frac{2}{q+\sqrt{q^2+4p}} \text{ as } n \longrightarrow -\infty .$$

In this time, we have the following

THEOREM 2 : If $ru_1-u_2\neq 0$, then $\lim\limits_{n\to-\infty} \dfrac{u_n}{u_{n+1}} = \dfrac{2}{q-\sqrt{q^2+4p}}$;

if $ru_1-u_2=0$, then $\lim\limits_{n\to-\infty} \dfrac{u_n}{u_{n+1}} = \dfrac{2}{q+\sqrt{q^2+4p}}$

Now if we let $U=\begin{pmatrix} u_2 & u_1 \\ u_1 & u_0 \end{pmatrix}$, $V=\begin{pmatrix} q & 1 \\ p & 0 \end{pmatrix}$, where $u_0=\dfrac{u_2-qu_1}{p}$,

it can be proved by mathematical induction that for $n\in N$,

$$UV^n = \begin{pmatrix} u_{n+2} & u_{n+1} \\ u_{n+1} & u_n \end{pmatrix} \ \cdots\cdots\cdots\cdots(***)$$

If $u_2-su_1=0$, then

$$\frac{u_{n+1}}{u_n}=\frac{((r^n-s^n)u_2+(rs^n-r^ns)u_1)/(r-s)}{((r^{n-1}-s^{n-1})u_2+(rs^{n-1}-r^{n-1}s)u_1)/(r-s)}$$

$$=\frac{(r^n-s^n)su_1+(rs^n-r^ns)u_1}{(r^{n-1}-s^{n-1})su_1+(rs^{n-1}-r^{n-1}s)u_1}$$

$$=\frac{(rs^n-s^{n+1})u_1}{(rs^{n-1}-s^n)u_1}=s=\frac{q-\sqrt{q^2+4p}}{2}$$

Hence

$$\frac{u_{n+1}}{u_n}\to\frac{q-\sqrt{q^2+4p}}{2}\quad \text{as } n\to\infty.$$

Up to now, we have the following

THEOREM I : If $u_2-su_1\neq 0$, then $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=\dfrac{q+\sqrt{q^2+4p}}{2}$;

if $u_2-su_1=0$, then $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=\dfrac{q-\sqrt{q^2+4p}}{2}$

Now if $n\leq 0$, then from (*), we have that

$$\begin{aligned}(u_{n-1}-r^{-1}u_n) &= s^{-1}(u_n-r^{-1}u_{n+1}),\\ &= s^{-2}(u_{n+1}-r^{-1}u_{n+2})\\ &= \cdots\cdots\cdots\\ &= s^{n-2}(u_1-r^{-1}u_2)\end{aligned}$$

Therefore,

$$\begin{aligned}u_{n-1}-r^{-1}u_n &= s^{n-2}(u_1-r^{-1}u_2),\\ r^{-1}(u_n-r^{-1}u_{n+1}) &= r^{-1}s^{n-1}(u_1-r^{-1}u_2),\\ r^{-2}(u_{n+1}-r^{-1}u_{n+2}) &= r^{-2}s^n(u_1-r^{-1}u_2),\\ &\cdots\cdots\cdots\\ r^{n-1}(u_0-r^{-1}u_1) &= r^{n-1}s^{-1}(u_1-r^{-1}u_2).\end{aligned}$$

Adding the above equations, we obtain that we only have to take r, s to be the solutions of the equation $X^2-qX-p=0$

Without loss of generality, let $r=(q+\sqrt{q^2+4p})/2$, $s=(q-\sqrt{q^2+4p})/2$.

Now for positive integer n, we have from (*) that

$$\begin{aligned} u_{n+1}-r\,u_n &= s(u_n-r\,u_{n-1}) \\ &= s^2(u_{n-1}-r\,u_{n-2}) \\ &= \dots\dots\dots \\ &= s^{n-1}(u_2-ru_1) \end{aligned}$$

Hence,

$$\begin{aligned} u_{n+1}-r\,u_n &= s^{n-1}(u_2-ru_1) \\ r(u_n-ru_{n-1}) &= rs^{n-2}(u_2-ru_1) \\ r^2(u_{n-1}-ru_{n-2}) &= r^2s^{n-3}(u_2-ru_1) \\ &\dots\dots\dots\dots\dots\dots \\ r^{n-2}(u_3-ru_2) &= r^{n-2}s(u_2-ru_1) \;. \end{aligned}$$

Adding the above equations, we obtain that

$$u_{n+1}-r^{n-1}u_2=(s^{n-1}+rs^{n-2}+r^2s^{n-3}+\dots+r^{n-2}s)(u_2-ru_1)$$

or

$$u_{n+1}=\frac{r^n-s^n}{r-s}u_2+\frac{rs^n-r^ns}{r-s}u_1 \quad .$$

Now if $u_2-su_1\neq 0$, then since

$$0<\left|\frac{s}{r}\right|=\frac{(q-\sqrt{q^2+4p})/2}{(q+\sqrt{q^2+4p})/2}<1$$

thus $(s/r)^{n-1}\to 0$ as $n\to\infty$.

Hence

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{((r^n-s^n)u_2+(rs^n-r^ns)u_1)/(r-s)}{((r^{n-1}-s^{n-1})u_2+(rs^{n-1}-r^{n-1}s)u_1)/(r-s)} \\ &= \frac{(r-(s/r)^{n-1}s)u_2+((s/r)^{n-1}rs-rs)u_1}{(1-(s/r)^{n-1})u_2+((s/r)^{n-1}r-s)u_1} \\ &\longrightarrow \frac{ru_2-rsu_1}{u_2-su_1}=r=\frac{q+\sqrt{q^2+4p}}{2} \quad \text{as } n\longrightarrow\infty \end{aligned}$$

1. DEFINITIONS

In this paper, we want to extend the idea of the generalized Fibonacci sequence to a complicated way as the following:

Let $u_1=\alpha$, $u_2=\beta$ be two arbitrarily complex numbers not both zeros, let p and q be positive reals.

Define $u_{n+1}=pu_{n-1}+qu_n$ for $n \in Z$. It is clear that $u_{n+1}=f_{n+1}$ if $p=q=1$, where $f_1=\alpha$, $f_2=\beta$, and $f_{n+2}=f_n+f_{n+1}$ for $n \in Z$.[4].

We also define that the extended Fibonacci sequence $(u_n)_{n\in Z}$ is alternating if $u_{n+1}/u_n < 0$ for all $n \in S$, where $S=Z$ or $S=N$.

2. LEMMAS AND THEOREMS

LEMMA 1 : In the extended Fibonacci sequence $(u_n)_{n\in Z}$, $u_k = 0$ for at most one $k \in Z$.

PROOF : Suppose that $u_m=u_k=0$ for some $m>k$, then $u_{k-1}\neq 0$, for otherwise $u_n=0$ for all $n \in Z$, contradicts to the choice of u_1 and u_2. Now we see that $u_{k+1}=pu_{k-1}$, $u_{k+2}=pqu_{k-1}$, $u_{k+3}=(p^2+pq^2)u_{k-1}$,.........In general, $u_n=u_{k-1}$ times some positive real, hence in particular, $u_m=u_{k-1}$ times some positive real, this contradicts to the assumption that $u_m=0$. Therefore, $u_m=u_k=0$ for some $m>k$ is impossible. This completes the proof.

As a consequence of the above lemma, it is meaningful to consider

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} \qquad \text{and} \qquad \lim_{n\to-\infty} \frac{u_n}{u_{n+1}}$$

respectively.

Now in order for the defining equation " $u_{n+1}=pu_{n-1}+qu_n$ " to be of the form

$$(u_{n+1} - r\, u_n) = s(u_n - r\, u_{n-1})............(*)$$

省立新竹師範學院新竹師院第一期學報
民國七十六年第四〇二頁至第三九五頁

On Fibonacci Sequence

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ABSTRACT

In this paper, we want to consider some properties of the sequence defined by $u_1=\alpha$, $u_2=\beta$, $u_{n+1}=pu_{n-1}+qu_n$, where $n\in Z$, α and β be two arbitrarily complex numbers not both zeros, and P, q be positive reals.

The most remarkable results are:

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\frac{q+\sqrt{q^2+4p}}{2} \quad \text{if } \beta-\frac{q-\sqrt{q^2+4p}}{2}\alpha\neq 0$$

$$=\frac{q-\sqrt{q^2+4p}}{2} \quad \text{if } \beta-\frac{q-\sqrt{q^2+4p}}{2}\alpha=0$$

$$\lim_{n\to-\infty}\frac{u_n}{u_{n+1}}=\frac{2}{q-\sqrt{q^2+4p}} \quad \text{if } \beta-\frac{q+\sqrt{q^2+4p}}{2}\alpha\neq 0$$

$$=\frac{2}{q+\sqrt{q^2+4p}} \quad \text{if } \beta-\frac{q+\sqrt{q^2+4P}}{2}\alpha=0$$